

DYNAMICAL RIGIDITY OF NON DISCRETE REPRESENTATIONS IN $\mathrm{PSL}(2, \mathbb{R})$

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ABSTRACT. The aim of this note is to advertise on a result, not stated explicitly, but proved, in [Wo1]. Namely, if Γ is any group, if ρ_1, ρ_2 are representations of Γ in $\mathrm{PSL}(2, \mathbb{R})$, one of them being non elementary and non discrete, and if for all $\gamma \in \Gamma$, $\rho_1(\gamma)$ and $\rho_2(\gamma)$ have the same rotation number, then ρ_1 and ρ_2 are conjugate in $\mathrm{PSL}(2, \mathbb{R})$. In particular, if two non discrete, non elementary representations yield semi-conjugate actions on the circle, then they are conjugate in $\mathrm{PSL}(2, \mathbb{R})$.

Recall that the group $\mathrm{PSL}(2, \mathbb{R})$ of orientation-preserving isometries of the hyperbolic plane \mathbb{H}^2 may be considered as a subgroup of the group $\mathrm{Homeo}_+(S^1)$ of orientation-preserving homeomorphisms of the circle (the circle at infinity of \mathbb{H}^2). Any element of $\mathrm{Homeo}_+(S^1)$ has a rotation number, invariant under semi-conjugation (see eg [Gh2, Ma] for definitions, and a lot of information). If $A \in \mathrm{PSL}(2, \mathbb{R})$, its rotation number, $\mathrm{rot}(A) \in \mathbb{R}/2\pi\mathbb{Z}$ is equal to zero if A is hyperbolic, parabolic or the identity, and to its angle of rotation if A is elliptic.

If Γ is a fundamental group of a compact orientable surface, any Teichmüller representation ρ of Γ in $\mathrm{PSL}(2, \mathbb{R})$ admits continuous deformations (in the Teichmüller space), which are C^0 -conjugate, but not even C^1 -conjugate to ρ (see [Gh1], Proposition III.4.1). More generally, if Γ is any finitely generated group, then every non-elementary, discrete representation ρ of Γ in $\mathrm{PSL}(2, \mathbb{R})$ may be deformed to representations C^0 -conjugate to ρ , but not conjugate in $\mathrm{PSL}(2, \mathbb{R})$, by deforming the Fuchsian group $\rho(\Gamma)$. In contrast, we may observe the following.

Proposition 1. *Let Γ be any group, and $\rho_1, \rho_2 \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$. Suppose ρ_1 is non elementary, non discrete, and suppose that for every $\gamma \in \Gamma$, $\mathrm{rot}(\rho_1(\gamma)) = \mathrm{rot}(\rho_2(\gamma))$. Then there exists $g \in \mathrm{PSL}(2, \mathbb{R})$ such that $\rho_1 = g\rho_2g^{-1}$.*

In the statement above, a representation ρ is called elementary if $\rho(\Gamma)$ is elementary, ie, has a finite orbit in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ (see eg [Ka], section 2.4).

Although Proposition 1 was proved in [Wo1] (see Theorem 2.39 in [Wo1]), it was not stated explicitly there, nor mentionned in the introduction. It was even not included in the published version, [Wo2]. The aim of this note, thus, is to make this statement more accessible. It is motivated, in part, by the recent article [KKM], as Proposition 1 answers a part of Question 1.13 of [KKM].

Even though the proof of Proposition 1 is already in [Wo1], it is short and elementary so I reproduce it here.

Proof. By Selberg's lemma, the subgroup $\rho_1(\Gamma)$ of $\mathrm{PSL}(2, \mathbb{R})$ admits a torsion-free finite index subgroup, which, thus, is still non elementary and non discrete. Hence, $\rho_1(\Gamma)$ admits an elliptic element of infinite order (see [Ka],

Theorem 2.4.5), say, $\rho_1(\gamma_0)$, for some $\gamma_0 \in \Gamma$. We may conjugate ρ_1 so that $\rho_1(\gamma_0) = \pm \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, with $\theta \in (0, \pi)$ irrational (indeed, this matrix is a rotation of angle 2θ in \mathbb{H}^2). Also, $\rho_2(\gamma_0)$, having rotation number 2θ , is elliptic, conjugate to $\rho_1(\gamma_0)$. Up to conjugating ρ_2 , we may further suppose that $\rho_1(\gamma_0) = \rho_2(\gamma_0)$.

Now let $\gamma \in \Gamma$. For $i \in \{1, 2\}$, write $\rho_i(\gamma) = \pm \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. Then for all $n \in \mathbb{Z}$,

$$|\mathrm{tr} \rho_i(\gamma \gamma_0^n)| = |(a_i + d_i) \cos(n\theta) + (c_i - b_i) \sin(n\theta)|;$$

this is the absolute value of the scalar product of the vectors $(a_i + d_i, c_i - b_i)$ and $(\cos(n\theta), \sin(n\theta))$ of \mathbb{R}^2 . The first vector cannot be zero, and the second vector may be chosen as close as we want from any point of the unit circle. Thus, for infinitely many values of n , the quantity $|\mathrm{tr}(\rho_1(\gamma \gamma_0^n))|$ is in $(0, 2)$, hence $\rho_1(\gamma \gamma_0^n)$ is elliptic, thus, so is $\rho_2(\gamma \gamma_0^n)$, and the cosines of their rotation numbers are equal, hence

$$|(a_1 + d_1) \cos(n\theta) + (c_1 - b_1) \sin(n\theta)| = |(a_2 + d_2) \cos(n\theta) + (c_2 - b_2) \sin(n\theta)|.$$

This equality, valid for infinitely many distinct values of n , implies that $|a_1 + d_1| = |a_2 + d_2|$. This proves that $|\mathrm{tr}(\rho_1(\gamma))| = |\mathrm{tr}(\rho_2(\gamma))|$, for all $\gamma \in \Gamma$.

It is well known that, if the traces of two irreducible representations in $\mathrm{SL}(2, \mathbb{C})$ agree on each element of the group, then they are conjugate, see [CS], Proposition 1.5.2. An adaptation of this argument (see eg [Wo2], Proposition 2.15) gives, in our context, that ρ_1 and ρ_2 are conjugated by an element g of $\mathrm{Isom}(\mathbb{H}^2)$. This g preserves the orientation of the hyperbolic plane, otherwise we would have $\mathrm{rot}(\rho_1(\gamma_0)) = -\mathrm{rot}(\rho_2(\gamma_0))$. Finally, $\rho_2 = g\rho_1g^{-1}$, with $g \in \mathrm{PSL}(2, \mathbb{R})$. \square

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